

# Some Theorems on Optimality of a Single Observation Confidence Interval for the Mean of a Normal Distribution

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Dedicated to the memory of Charles Stein (1920 - 2016)

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## Abstract

We consider the problem of finding a proper confidence interval for the mean based on a single observation from a normal distribution with both mean and variance unknown. Portnoy (2017) characterizes the scale-sign invariant rules and shows that the Hunt-Stein construction provides a randomized invariant rule that improves on any given randomized rule in the sense that it has greater minimal coverage among all procedures with a fixed expected length. Mathematical results here provide a specific mixture of two non-randomized invariant rules that achieve the minimax optimality. A multivariate confidence set based on a single observation vector is also developed.

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# 1 Introduction and basic result

Consider a single observation  $X \sim \mathcal{N}(\mu, \sigma^2)$ . Let  $\lambda = \mu/\sigma$  and note that  $X/\sigma \sim \mathcal{N}(\lambda, 1)$ .

Now consider the following confidence intervals: let  $c_1 < c_2$  and define the interval

$$CI^* \equiv CI^*(X; c_1, c_2) = \begin{cases} c_1 X \leq \mu \leq c_2 X & X > 0 \\ c_2 X \leq \mu \leq c_1 X & X < 0 \end{cases} \quad (1)$$

Portnoy (2017) provided the following coverage formula:

**Theorem 1** *The probability of coverage for the interval,  $CI^*$  for  $\lambda > 0$  is:*

$$P(\lambda; c_1, c_2) = \begin{cases} \Phi\left(\lambda\left(1 - \frac{1}{c_2}\right)\right) + 1 - \Phi\left(\lambda\left(1 + \frac{1}{c_1}\right)\right) & c_1 \leq 0; c_2 \geq 0 \\ \Phi\left(\lambda\left(1 - \frac{1}{c_2}\right)\right) - \Phi\left(\lambda\left(1 + \frac{1}{c_1}\right)\right) & c_1 > 0; c_2 > 0 \end{cases} \quad (2)$$

*Note that the first line above holds for  $c_1 = 0$  and/or  $c_2 = 0$  by taking limits as  $c_1 \nearrow 0$  and/or  $c_2 \searrow 0$ . The coverage probability for other cases is given from these results by symmetry.*

Portnoy (2017) also characterizes the scale-sign invariant rules as having the form of  $CI^*$  and provides a version of the Hunt-Stein Theorem (Hunt and Stein, 1945, also see Lehmann, 1959) to show that for any (randomized) confidence interval, there is a randomized invariant rule whose minimal coverage (over the parameters) is larger and whose expected length is the same. Section 2 below finds a specific mixture of two non-randomized invariant rules that achieves minimaxity (in the sense that it maximizes minimal coverage among all rules with fixed expected length). Section 3 provides a brief discussion of numeric computation. Section 4 proves that there is a norm-bounded confidence set that provides a proper confidence set for the

mean based on a single (multivariate) observation from a multivariate normal distribution with arbitrary mean and covariance matrix.

## 2 Optimal Mixture

The first rather complicated theorem shows that for any randomized invariant procedure there is a mixture of no more than 8 specific non-randomized invariant confidence intervals that is as good (in the minimax sense above). A corollarys use linear programming theory to show that a mixture of two specific intervals suffices. It also shows that there is a best such rule, and clearly this rule must be minimax (since no other rule can be strictly better).

**Theorem 2** *Let  $F$  be a probability distribution on  $\{c_1 < c_2\}$  generating a randomized invariant confidence interval. Then there are constants:  $c_1^* \leq a_1^* \leq 0 < 1 \leq c_2^*$  and a finite mixture,  $F^*$ , on the intervals:  $[c_1^*, 1]$ ,  $[a_1^*, 1]$ ,  $[c_1^*, c_2^*]$ ,  $[a_1^*, c_2^*]$ ,  $[0, 1]$ ,  $[0, c_2^*]$ ,  $[1, c_2^*]$ , and  $\phi$  (the empty interval) with at least as large minimal coverage probability and no larger expected length. That is,*

$$\begin{aligned} \inf_{\lambda} E_{F^*} P(\lambda; C_1, C_2) &\geq \inf_{\lambda} E_F P(\lambda; C_1, C_2) \\ E_{F^*}(C_2 - C_1) &\leq E_F(C_2 - C_1) , \end{aligned}$$

where  $P$  denotes the coverage probability given by (2) and is repeated below for convenience.

**Proof.** The proof is given by a series of lemmas. To simplify notation, refer to the interval  $CI^*(X; c_1, c_2)$  as  $[c_1, c_2]$ . By scale and sign invariance, we can restrict to the case  $\lambda \geq 0$  without loss of generality.

**Lemma 1** *The distribution  $F$  can be restricted to one putting probability 1 on the set  $\{[c_1, c_2] : -c_2 \leq c_1 \leq 1 \text{ and } c_2 > 0\}$ .*

**Proof.** To show that we can take  $c_2 \geq 0$ , set  $c_1 = c_2 - h$ , use Theorem 1, and consider

$$\frac{\partial}{\partial c_2} P(\lambda; c_2 - h, c_2) = \frac{\lambda}{c_2^2} \varphi \left( \lambda \left( 1 - \frac{1}{c_2} \right) \right) - \frac{\lambda}{(c_2 - h)^2} \varphi \left( \lambda \left( 1 - \frac{1}{c_2 - h} \right) \right).$$

For  $c_2 < 0$ , both factors of the first summand above are greater than the corresponding factors of the second summand, and so it follows that the function  $P(\lambda; c_2 - h, c_2) > 0$  is increasing in  $c_2$ . Therefore, the interval  $[-(c_2 - c_1), 0]$  has the same length but larger probability than  $[c_1, c_2]$ . So we can take  $c_2 \geq 0$ . To show the inequality is strict, we have (from Theorem 1)

$$\begin{aligned} P(\lambda, c_1, 0) &= 1 - \Phi \left( \lambda \left( 1 - \frac{1}{c_1} \right) \right) = \Phi \left( \lambda \left( \frac{1}{c_1} - 1 \right) \right) \\ &< \Phi \left( \lambda \left( 1 + \frac{1}{c_1} \right) \right) = P(\lambda, 0, -c_1), \end{aligned}$$

and so  $c_2$  can be taken to be strictly positive.

A similar proof shows that we can take  $c_1 \leq 1$ : let  $c_2 = c_1 + h$  and consider

$$\frac{\partial}{\partial c_1} P(\lambda; c_1, c_1 + h) = \frac{\lambda}{(c_1 + h)^2} \varphi \left( \lambda \left( 1 - \frac{1}{c_1 + h} \right) \right) - \frac{\lambda}{c_1^2} \varphi \left( \lambda \left( 1 - \frac{1}{c_1} \right) \right).$$

For  $c_1 \geq 1$ , both factors of the first summand above are smaller than the corresponding factors of the second summand, and so it follows that the function  $P(\lambda; c_1, c_1 + h) < 0$  is decreasing; and so the interval  $[1, c_2 - c_1 + 1]$  has the same length and larger probability than  $[c_1, c_2]$ .

Finally, to show that we can take  $c_1 > -c_2$ , first note that if  $c_1 \geq 0$ , the inequality is immediate (since  $c_2 > 0$ ). Next, to show that if this inequality fails, the interval  $[-c_2, -c_1]$  has larger probability (and the same length) as

$[c_1, c_2]$ , define  $\Delta \equiv P(\lambda; c_1, c_2) - P(\lambda; -c_2, c_1)$ . Let  $b_1 = 1/c_1$ ,  $b_2 = 1/c_2$ , and define  $h$  so that  $b_2 = -b_1 - h$ . Note that  $b_2 < -b_1$  (or equivalently,  $c_2 > -c_1$  if and only if  $h > 0$ ). Then,

$$\Delta = \Phi(\lambda(1 + b_1 + h)) - \Phi(\lambda(1 - b_1)) - \Phi(\lambda(1 + b_1)) + \Phi(\lambda(1 - b_1 + h))$$

and

$$\frac{\partial}{\partial h} \Delta = \lambda \varphi(\lambda(1 + b_1 + h)) - \varphi(\lambda(1 - b_1 + h)) > 0.$$

for  $h > 0$ . Now  $\Delta = 0$  when  $h = 0$ ; and hence  $\Delta \geq 0$  as long as  $h \geq 0$ . Therefore, the interval  $[c_1, c_2]$  has larger coverage probability than  $[-c_2, -c_1]$  as long as  $h \geq 0$ , or equivalently  $c_2 > -c_1$ . ■

The following Lemma presents some derivative calculations and subsequent convexity and concavity properties that will facilitate analyzing the coverage probabilities.

**Lemma 2**

$$\frac{\partial^2 \Phi(\lambda(1 - 1/c))}{\partial c^2} = \frac{\lambda}{c^3} \varphi\left(\lambda\left(1 - \frac{1}{c}\right)\right) \left[\frac{\lambda^2}{c^2} - \frac{\lambda^2}{c} - 2\right] \quad (3)$$

$$\frac{\partial \Phi(\lambda d)}{\partial d} = d \varphi(d \lambda), \quad (4)$$

$$\frac{\partial^2 \Phi(\lambda d)}{\partial d^2} = -\lambda d^3 \varphi(d \lambda) \quad (5)$$

From (3), there are functions  $a_1(\lambda) < 0$  and  $a_2(\lambda) > 1$  such that the coverage probability  $P(\lambda; c_1, c_2)$  is concave in  $c_1$  for  $c_1 \leq a_1(\lambda)$ , convex in  $c_1$  for  $a_1(\lambda) \leq c_1 \leq 0$ , and a possibly different convex function for  $0 \leq c_1 \leq 1$ ; and is convex in  $c_2$  for  $0 \leq c_2 \leq a_2(\lambda)$  and convex in  $c_2$  for  $c_2 \geq a_2(\lambda)$ .

From (5),  $\Phi(\lambda d)$  is increasing and concave in  $\lambda$  for  $d \geq 0$ , and decreasing and convex in  $\lambda$  for  $d \leq 0$ .

**Proof.** The derivative calculations are straightforward, using the fact that  $\varphi'(x) = -x\varphi(x)$ . Convexity and concavity in  $\lambda$  is also a trivial consequence of (5).

For the behavior of the coverage probability as a functions of  $c_1$  and  $c_2$ , note that derivatives of  $P(\lambda; c_1, c_2)$  will have the form (3) (with arguments  $c_1$  or  $c_2$ ). Clearly (3) vanishes if (and only if)

$$\frac{1}{c} = \frac{1 \pm \sqrt{1 + 8/\lambda^2}}{2} \equiv \{a_1(\lambda) < 0, a_2(\lambda) > 1\} \quad (6)$$

Thus  $P(\lambda; c_1, c_2)$  has sign changes only at  $a_1(\lambda) < 0$  and  $a_2(\lambda) > 1$ . The convexity and concavity claims follow directly by examining the behavior of  $P(\lambda; c_1, c_2)$  as  $c_1$  and  $c_2$  tend to  $\infty$ , 0, 1, 0, and  $-\infty$ , and noting that the derivatives are discontinuous at  $\lambda = 0$ . ■

**Lemma 3** *Given any distribution,  $F(c_1, c_2)$ , generating a randomized invariant confidence interval, there is a constant  $c_2^*$ , a random variable,  $C \sim F_1$  where  $F_1$  is a distribution on  $(-\infty, 0)$ , and a family of conditional distributions,  $G_c(c_1, c_2)$ , such that conditional on  $C = c$ ,  $G_c$  is finite discrete mixture on the intervals:  $[c, 1]$ ,  $[c, c_2^*]$ ,  $[0, 1]$ ,  $[0, c_2^*]$ ,  $[1, c_2^*]$ , and  $\phi$  (the empty interval); and such that the randomized confidence interval given by  $F_1$  and  $G_c$  has coverage probability no smaller than that of  $F$  and expected length no larger that of  $F$  uniformly in  $\lambda$ . Furthermore, the improvement is strict unless  $G_c$  is such a finite mixture. Finally, for each fixed  $\lambda$  there are functions  $a_1(\lambda)$  (see (6)) and  $c_1^*(\lambda) < a_1(\lambda)$  such that  $F_1$  can be replaced by a finite discrete mixture on  $\{c_1^*(\lambda), a_1(\lambda), 0\}$  giving no smaller coverage probability and no larger expected length at the specific value of  $\lambda$ . That is, “ $c$ ” in the first two intervals in the list above can be replaced by either  $c_1^*(\lambda)$  or  $a_1(\lambda)$  to provide a list of 8 intervals. Again, the improvement is strict unless  $F_1$  is such a mixture.*

**Proof.** Part 1:  $c_2 \geq 1$ . Fix the lower endpoint,  $c_1$  and let  $d \equiv (1 - 1/c_2)$ . Then  $d \in [0, 1]$  and from Lemma 2 (see (5)), the second derivative of the coverage probability is just  $E_F[-\lambda D^3 \varphi(Dx)]$ ; and so the coverage probability is (strictly) concave. Also,  $c_2 = 1/(1 - d)$  is convex (for  $d \geq 1$ ). Thus, from Jensen’s inequality, any  $F$ -probability on  $c_2 > 1$  can be replaced

by a point mass at  $c_2^* = 1/(1 - d^*)$  where  $d^* = ED \in (0, 1)$  for which both

$$P(\lambda, c_1, c_2^*) > E_F[P(\lambda, c_1, C) \mid C > 1] \quad \text{and} \quad c_2^* \leq E[C \mid C > 1]$$

uniformly in  $\lambda$ .

Part 2a:  $0 < c_2 \leq 1$  and  $0 \leq c_1 \leq 1$ . From Lemma 2 (see (3)), the coverage probability is convex on  $c_1 \leq c_2 \leq 1$ . So choose  $q$  so that

$$1 - q = E_F[P(C_2, \cdot | \lambda) \mid c_1 \leq C_2 \leq 1] .$$

. Then the mixture taking  $C_2 = c_1$  with probability  $q$  and  $C_2 = 1$  with probability  $1 - q$  generates the empty interval,  $\phi$ , (with probability  $q$ ) and the interval  $[c_1, 1]$  (with probability  $1 - q$ ) satisfying (simultaneously and uniformly in  $\lambda$ ) both

$$\begin{aligned} q P(\lambda; \phi) + (1 - q) P(\lambda; c_1, 1) &> E_F[P(\lambda; c_1, C_2) \mid c_1 \leq C_2 \leq 1] \\ q \times 0 + (1 - q) \times (1 - c_1) &= E_F[C_2 \mid c_1 \leq C_2 \leq 1] \end{aligned}$$

Part 2b:  $0 \leq c_2 \leq 1$  and  $-\infty < c_1 < 0$ . Again, the coverage probability is convex, and the probability on  $0 \leq C_2 \leq 1$  can be replaced by a mixture on 0 and 1 (with corresponding intervals:  $[c_1, 0]$  and  $[c_1, 1]$ ). This provides the first part of the Lemma.

Part 3: Finally, to replace  $F_1$  by a finite discrete mixture, note that (as above) the coverage probability is convex on  $[a_1(\lambda), 0]$  and concave on  $[-\infty, a_1(\lambda)]$ . Thus, probability on  $[a_1(\lambda), 0]$  can be replaced by a mixture on  $a_1(\lambda)$  and 0 having larger coverage probability and the same (conditional) expected length. By Jensen's inequality. and probability on  $[-\infty, a_1(\lambda)]$  can be replaced by a point mass at  $c_1^* \equiv E[C_1 \mid -\infty \leq C_1 \leq a_1(\lambda)]$  with larger

coverage probability (and the same conditional expected length). Note that, since  $c_1 > -c_2^*$ , if  $a_1(\lambda) \leq -c_2^*$  then the last interval is empty, and no point mass at  $c_1^*$  is needed. ■

To complete the proof of the Theorem, let  $F^*$  denote the distribution generated by  $F_1$  and  $G_c$  given by the first part of Lemma3, and let  $F_\lambda$  denote the discrete mixture given by the last part of the Lemma. Let  $C \sim F_1$  under  $F^*$  and have the two-point mixture on  $c_1^*(\lambda)$  and  $a_1(\lambda)$  under  $F_\lambda$ . Since  $F_\lambda$  improves only at a fixed  $\lambda$  it remains to find a rule where the improvement is uniform in  $\lambda$ .

From (2) the coverage probability (under  $F^*$  and  $F_\lambda$ ) is a linear combinations of functions  $\Phi(\lambda(1 - 1/c))$  where  $c$  is  $C$  or 0 or 1 or  $c_2^*$ . For  $c = 0$  or  $c = 1$ , the function is constant (in  $\lambda$ ), and so the mixture probabilities will sum to provide the coverage probability of the form

$$P(\lambda) = b_0 + b_1 \Phi\left(\lambda\left(1 - \frac{1}{d_2^*}\right)\right) - b_2 E\Phi\left(\lambda\left(1 - \frac{1}{C}\right)\right) \quad (7)$$

where the expectation is under  $F^*$  or  $F_\lambda$ . Note that the coefficients,  $b_i$ , are non-negative and are exactly the same under  $F^*$  and  $F_\lambda$  (since the probability that  $c < 0$  is the same under each distribution). Note that  $(1 - 1/c_2^*) < 1$  (since  $c_2^* > 1$ ) and  $(1 - 1/C) > 1$  (since  $C < 0$ ). Thus, from (4) the  $\lambda$ -derivative of  $P$  (see (7)) becomes

$$\begin{aligned} P'(\lambda) &= b'_1 \varphi\left(\lambda\left(1 - \frac{1}{d_2^*}\right)\right) - b_2 E\left(1 - \frac{1}{C}\right) \varphi\left(\lambda\left(1 - \frac{1}{C}\right)\right) \\ &= \varphi(0) \left\{ b'_1 - b_2 E\left(1 - \frac{1}{C}\right) \exp\left(-\frac{1}{2}\lambda^2 \left[\left(1 - \frac{1}{C}\right)^2 - \left(1 - \frac{1}{c_2^*}\right)^2\right]\right) \right\}. \end{aligned}$$

Note that the coefficient of  $\lambda^2$  in the exponential function is strictly positive, and so the exponential function is monotonically (strictly) decreasing to zero.



It follows (by the monotone convergence theorem) that  $P'(\lambda)$  is the difference between a positive constant and a function that is decreasing monotonically to zero. Thus  $P(\lambda)$  is monotonically increasing and is positive for  $\lambda$  large. So  $P(\lambda)$  can not be minimized as  $\lambda \rightarrow \infty$ . If  $P'(0) \geq 0$ ,  $P(\lambda)$  is increasing, and thus minimized at  $\lambda = 0$ . Otherwise,  $P(\lambda)$  has a unique minimum at  $\lambda = \lambda^* \in (0, \infty)$ .

Case 1:  $P'_{F^*}(0) > 0$  and  $P_{F^*}(\lambda^*)$  is minimized at  $\lambda = 0$ .

As  $\lambda \rightarrow 0$ ,  $a_1(\lambda) \rightarrow 0$ , and so  $F_\lambda$  tends to the distribution  $F_{\lambda=0}$  that puts all its probability at the point  $\tilde{c}_1 = E_{F^*}[C | C < 0]$ . By dominated convergence (and (7)),  $P_{F_\lambda}(\lambda) \rightarrow P_{F_{\lambda=0}}(0)$ . Furthermore, since the expected length is the same for all  $F_\lambda$ ,  $F_{\lambda=0}$  also has the same expected length as  $F^*$ . As noted above, both  $P_{F^*}(0)$  and  $P_{F_{\lambda=0}}(0)$  are monotonically increasing, and so both are minimized at  $\lambda = 0$ . So the interval defined using  $F_{\lambda=0}$  is at least as good as that defined using  $F^*$ .

Case 2:  $P'_{F^*}(0) < 0$  and  $P_{F^*}(\lambda^*)$  is a unique minimum.

Consider small interval around  $\lambda^*$ . If  $F^*$  not in the family of mixtures, coverage for  $F_{\lambda^*}$  is strictly uniformly greater by  $\delta > 0$  on the interval.

Since coverage is bounded above by 1, one can choose  $\epsilon$  (depending only on  $\delta$ ) small enough that  $G_\epsilon \equiv (1 - \epsilon)F^* + \epsilon F_{\lambda^*}$  satisfies:

$$\inf_{\lambda} E_{G_\epsilon} P(\lambda; C_1, C_2) > \inf_{\lambda} E_{F^*} P(\lambda; C_1, C_2) + \epsilon\delta/2.$$

See Figure 1. So  $F_0$  can not be minimax for  $\eta < \epsilon\delta/2$  except as a mixture of above form.

There is one remaining issue. The mixture used in the proof above included probability mass at  $c_2 = 0$ , while the statement of the Theorem omits

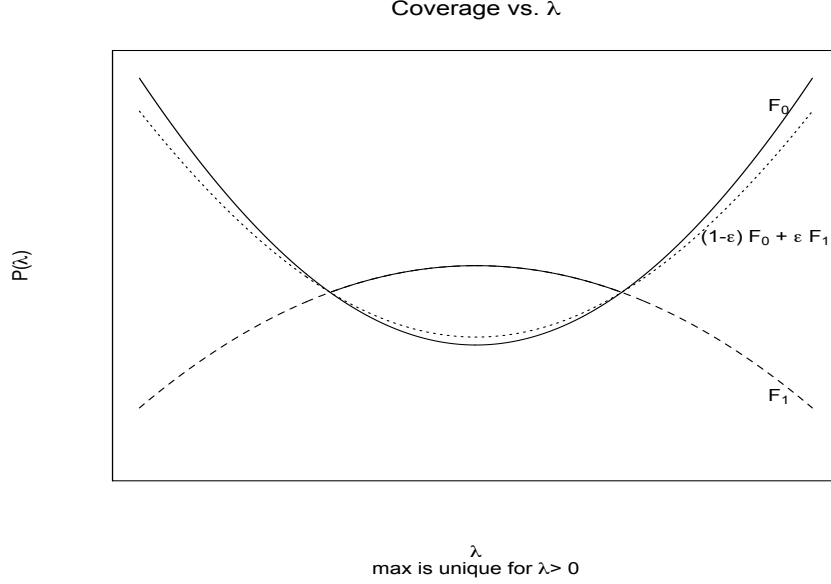


Figure 1: Mixture uniformly better than assumed minimax rule.

such mass. To complete the proof, use Lemma 1 to replace mass on intervals of the form  $[c_1, p]$  (with  $c_1 < 0$ ) by intervals  $[0, -c_1]$ . Then using the transformed mass to redefine  $c_2^*$  and the probabilities on the intervals  $\phi$  and  $[0, c_2^*]$ , the new distribution will provide a mixture where the only interval with its right endpoint equal to zero is  $[0, 0]$ , which is equivalent to the empty interval,  $\phi$ . ■

**Corollary 1** *Given  $h > 0$ , there is a mixture of two of the 8 intervals in Lemma 3 that is optimal in the minimax sense. From computational results described below, there are constants  $c_1 < a_1 \leq 0$  and a probability  $p \in [0, 1]$  such that the  $p$ -mixture of  $[c_1, c_2]$  and  $[a_1, c_2]$  is numerically “minimax”, where  $c_2$  is chosen so that the mixture has length  $h$  (that is,  $c_2$  satisfies  $h = p(c_2 - c_1) + (1 - p)(c_2 - a_1)$ ). Specifically, this two-point mixture numerically maximizes the minimal coverage probability (over  $\lambda$ ) among all rules with expected length  $h$ .*

**Proof.** Consider any mixture of the 8 intervals given in Theorem 1, and recall from the proof of the Theorem that the minimum coverage over  $\lambda$  occurs at a fixed value  $\lambda_0$  where  $\lambda_0 = 0$  or is the minimizing  $\lambda$ -value. Consider fixing the interval end points (say,  $\{(r_i, s_i) : i = 1, \dots, 8\}$ ). Then, as a function of the mixing probabilities  $(p_1, \dots, p_8)$ , the coverage probability is  $\sum_{i=1}^8 p_i P(\lambda_0, r_i, s_i)$  and the expected length is  $\sum_{i=1}^8 p_i (s_i - r_i)$ . Thus both the coverage probability and the expected length are linear in the  $p_i$ 's. Therefore, maximizing the coverage probability over  $(p_1, \dots, p_8)$  subject to  $\sum_{i=1}^8 p_i = 1$  and  $\sum_{i=1}^8 p_i (s_i - r_i) = h$  is a linear programming problem. As a consequence the coverage is maximized at a solution with at most two  $p_i$ 's non-zero.

Thus, an optimal rule can be found by considering each pair of the 8 intervals in Theorem 2 and optimizing over the endpoints and mixing probability. In examining the 8-choose-2 intervals, many have the same form or can be obtained from others by taking limits of the endpoints or the probability. Also, for the length  $h \leq 1$ , the intervals are of the form  $[c_1, 1]$  with  $c_1 < 0$ , all of which have coverage equal to .5. Thus,  $c_1 = 0$  minimizes the length, and by convexity (Lemma 3), the optimal rule for  $h \leq 1$  is a mixture of the interval  $[0, 1]$  and the “empty” interval,  $\phi$  (or equivalently,  $[0, 0]$ ), with mixing probability  $p = h$ ). As a consequence, only the following cases need to be treated (with the equivalent or redundant cases listed as “subcases”):

Case 1:  $[a_1, 1] [c_1, c_2] \quad -1 \leq a_1 < 0, -c_2 < c_1 < 0, c_2 > 1$

subcases:  $[c_1, 1] [c_1, c_2], [c_1, 1] [a_1, c_2], [a_1, 1] [a_1, c_2], [0, 1] [0, c_2]$   
 $[a_1, 1] [0, c_2], [c_1, 1] [0, c_2], [0, 1] [c_1, c_2], [0, 1] [a_1, c_2]$

Case 2:  $[a_1, c_2] [c_1, c_2]$   $c_1 < a_1 < 0$ ,  $-c_2 < c_1 < 0$ ,  $c_2 > 1$

subcases:  $[0, c_2] [c_1, c_2]$ ,  $[0, c_2] [a_1, c_2]$

Case 3:  $[a_1, 1] [0, c_2]$   $-c_2 < a_1 < 0$ ,  $c_2 > 1$

subcases:  $[c_1, 1] [0, c_2]$ ,  $[0, 1] [0, c_2]$ ,  $[a_1, 1] [0, 1]$ ,  $[c_1, 1] [0, 1]$

Case 4:  $[a_1, 1] [1, c_2]$   $-c_2 < a_1 < 0$ ,  $c_2 > 1$

coverage  $\leq .5$ , use  $\phi [0, 1]$

subcases:  $[c_1, 1] [1, c_2]$ ,  $[0, 1] [1, c_2]$

Case 5:  $[a_1, 1] [c_1, 1]$   $-1 \leq a_1 < 0$ ,  $a_1 < c_1 < 0$

coverage  $\leq .5$ , use  $\phi [0, 1]$

Case 6:  $\phi [a_1, 1]$   $-1 \leq a_1 < 0$

coverage  $\leq .5$ , use  $\phi [0, 1]$

subcase:  $\phi [c_1, 1]$

Case 7:  $\phi [c_1, c_2]$   $-c_2 < c_1 < 0, c_2 > 1$

subcase:  $\phi [a_1, c_2]$

Case 8:  $\phi [0, c_2]$   $c_2 > \text{coverage} \leq .5$ , use  $\phi [0, 1]$

subcases:  $\phi [1, c_2]$

Thus, only cases 1, 2, 3, and 7 need to be treated. Numerical optimization (discussed in Section 2) indicates that Case 2 is always at least as good as any other. In fact, as noted in Portnoy (2017), it appears that for  $h$  larger than a cutoff slightly less than 5 (coverage probability about .8), the mixing probability is 1, and the non-randomized invariant interval  $[c_1, c_2]$  is optimal, at least according to the numerical results.

However, as indicated in Section 3, the numeric optimization is surprisingly difficult, and can not prove that a given rule is optimal, or even that there is an optimal rule (as the minimax coverage may be a limit as the endpoints or probabilities tend to their boundaries). Therefore, it remains to prove that there is an optimal invariant mixture.

By the above proof of this Corollary, we need only show that each of the two-point mixtures in Cases 1, 2, 3, and 7 achieve the maximum of the minimum coverage (over  $\lambda$ ) at finite values for the endpoints and probability. Consider the apparent optimal rule given by Case 2. Since the minimizing  $\lambda^*$  is finite ( $\lambda^* = 0$  or  $\lambda^* \in (0, +\infty)$ ), the minimal coverage probability is continuous in  $(c_1, a_1, p)$ . Thus, the maximum will be attained as long as  $c_1$  is bounded away from  $-\infty$  (since  $c_1 < a_1 < 0$ , and  $p \in [0, 1]$ ). Since the length is fixed, if  $c_1$  were unbounded, then  $p$  would need to tend to zero (along some sequence). So consider the derivative of the coverage probability

for Case 2 as  $p \rightarrow 0$ . Using (2), the coverage probability becomes

$$p P(\lambda, c_1, c_2) + (1 - p) P(\lambda, a_1, c_2) = \\ \Phi \left( \lambda \left( 1 + \frac{1}{c_2} \right) \right) + 1 - p \Phi \left( \lambda \left( 1 + \frac{1}{c_1} \right) \right) - (1 - p) \Phi \left( \lambda \left( 1 + \frac{1}{a_1} \right) \right) .$$

Now  $h = p(c_2 - c_1) + (1 - p)(c_2 - a_1)$ , or  $c_2 = h + a_1 + p(c_1 - a_1)$ . Inserting  $c_2$  in the first term in (8) and differentiating with respect to  $p$  gives:

$$\frac{\lambda^*(c_1 - a_1)}{(h + a_1 + p(c_1 - a_1))^2} \varphi \left( \lambda \left( 1 + \frac{1}{c_2} \right) \right) - \Phi \left( \lambda \left( 1 + \frac{1}{c_1} \right) \right) + \Phi \left( \lambda \left( 1 + \frac{1}{a_1} \right) \right) .$$

The first term is clearly positive, and the difference in the last two is positive since  $c_1 < a_1$ . Thus, the minimal coverage probability can *not* be maximized as  $p \rightarrow 0$ . Hence, the maximum is attained at finite values. Entirely similar proofs work for the other 3 cases. As a consequence, from Theorem 1 and the Corollary, the optimal invariant rule is optimal among all rules. ■

### 3 Some details of the numerical optimization

Numerical optimization for each of the Cases above appears to be remarkably difficult and complicated. One minor complication is that a separate minimization over  $\lambda$  is needed before the minimal coverage can be maximized over the endpoints and probability variables. Fortunately, the R-function `optimize` (see R Core Team (2015)) appears to work quickly and efficiently for the  $\lambda$ -minimization, especially since it is possible to compute an upper bound on  $\lambda$  above which the  $\lambda$ -derivative is positive (and so which bounds the minimizing value).

Now consider numerically maximizing the minimal probability over the interval variables, say  $(c_1, a_1, c_2, p)$  for Case 2, subject to fixing

$$h = p(c_2 - c_1) + (1 - p)(c_2 - a_1) .$$

This presents a more serious problem: the coverage probability is not differentiable when any endpoint is zero. This suggests that trying to solve the equation of partial derivatives may be very problematic, thus precluding the use of Lagrange multipliers to handle the length constraint. As an alternative, solve the length equation for  $c_2$  and use the R-function `optim` (R Core Team (2015)) to maximize over  $(c_1, a_1, p)$ . Unfortunately, incorporating the constraints  $(-c_2 < c_1 < a_1 < 0, c_2 > 1)$  still posed numerical complications. This algorithm often worked, but for some  $h$ -values the routine indicated a failure to converge numerically, and in other cases gave very unreliable results depending on starting values used. Thus, an initial grid search was used (with mesh .1 in each variable), and the routine `optim` was used on the maximizing grid rectangle. Even then, some special programming was needed to deal with the constraints (especially for values of  $h$  less than 2.5). Nonetheless, after considerable refinement, the computer results appeared to be reliable, with accuracy of at least 4 decimal places. With the obvious modifications, the same code was used to treat the other cases. The output provided the plot in Portnoy (2017), though (of course) none of the numerical results can be guaranteed.

For completeness, the following gives the R-code used for Case 2 (omitting modification for smaller  $h$ -values):

```

# case 2 [c1,c2] [c11,c2] -c2 < c1 < 0 ; -c1 < c11 < 0
# b[1]=c1 , b[2] = c11
# b[3] = p ; h = p*(c2-c1) + (1-p)*(c2-c11)
c2h <- function(b,h) { p <- b[3]
return( h + p*b[1] + (1-p)*b[2] ) }
concheck <- function(c1,c11,c2) { # check constraints
return( (c2 > 1 & c1 > -c2 & c11 > c1) ) }
c1s <- -.00001 - .1*(0:200); c1s[11] <- -1
c11s <- -.00001 - .1*(0:200)
ps <- .1*(0:10) ; ps[11] <- .99999

P0 <- function(lam,b,h) { # b[1]= c1 , b[2] = c11 , b[3] = p
# h = p*(c2-b[1]) + (1-p)*(c2-b[2])
c2 <- c2h(b,h) ; p <- b[3]
return( p*P(lam,b[1],c2) + (1-p)*P(lam,b[2],c2) ) }

dP0 <- function(lam,b,h) { # b[1]= c1 , b[2] = c11 , b[3] = p
p <- b[3]
# h = p*(c2-b[1]) + (1-p)*(c2-b[2])
c2 <- c2h(b,h)
return( p*dP(lam,b[1],c2) + (1-p)*dP(lam,b[2],c2) ) }

P1 <- function(b1,h=h,ret=ret) {
# b1[1]=c1, b1[2]=c11, b1[3]=p ; min over lambda
# lam0 = new lam* on return

```



```

c2 <- c2h(b1,h) ; b <- c(b1,c2)
if(!(c2 > 1))
{if(ret) return(1+runif(1))
else return(list(objective=1+runif(1),min=-1)) }
# get upper bound lam0; from earlier runs lam0 = 3 should work,
# but check for P' > 0
d <- -1 ; lam0 = 2
while(d <= 0) {
d <- dP0(lam0,b,h)
lam0 <- lam0 + 1 }
# min over lam in [0, lam0]
m <- optimize(P0,c(0,lam0),b,h)
if(ret) return(m$objective) else return(m) }

for(i in 1:length(hs)) { h <- hs[i]
# max over grid
c1m <- 0 ; c11m <- 0 ; pm <- 0 ; Pm <- 0
for(c1 in c1s) { for(c11 in c11s) { for(p in ps) {
b <- c(c1,c11,p)
c2 <- c2h(b,h)
if( concheck(c1,c11,c2) ) {
Pn <- P1(b,h,T) # ; print(c(Pn,c1,c11,c2,p))
if(Pn > Pm) {
Pm <- Pn ; c1m <- c1 ; c11m <- c11 ; pm <- p } } } } }
# max over cell

```

```

c2 <- c2h(c(c1m,c11m,pm),h)
low <- c(max(c1m-.1,-c2),max(c11m-.1,c1),max(pm-.1,0))
up <- c(min(c1m+.1,-.000001),min(c11m+.1,-.000001),
min(pm+.1,.999999))
b0 <- (low+up)/2
m <- optim(b0,P1,lower=low,upper=up,
control=list(fnscale=-1),method='L-BFGS-B',h=h,ret=T)
if(m$converge != 0) print(paste(
"possible non-convergence", m$converge =,m$converge))
lam <- P1(m$par,h,F)
if(abs(lam$objective - m$value) > .000001)
print(paste( "lam min problem:"
"(P*,P(lam*)):",c(m$value,lam$objective)) )
out1[i,] <- c(m$value,c2h(m$par,h),m$par,h,lam$min) }
}

```

## 4 Multivariate Confidence Sets

**Theorem 3** *Let  $X \sim \mathcal{N}_p(\mu, \Sigma)$ . Then to achieve*

$$\inf_{\mu, \Sigma} P \{ \|\mu\| \leq c \|X\| \} \geq 1 - \alpha$$

*it suffices to take  $c = 3.85 \alpha^{-1/p}$ .*

**Proof.**

First (without loss of generality) assume  $\Sigma$  is non-singular (otherwise, restate the problem in a smaller dimensional space).

Now, let  $\Sigma = \Gamma' D(\gamma) \Gamma$  with  $\Gamma$  orthonormal, and let  $\gamma_0 = \min\{\gamma_i\}$ . Define  $\lambda = \gamma/\gamma_0$  and  $\nu = \mu/\gamma_0$ . Then (dividing through by  $\gamma_0$ ), the coverage probability is

$$CP = P \left\{ \|\nu\|^2 \leq c^2 \sum \lambda_i Y_i^2 \right\} \quad (8)$$

where  $Y_i$  are independent  $\mathcal{N}(\nu_i, 1)$ . Hence, from the well-known representation of a non-central Chi-square,  $Y_i^2 \sim \chi_{p+2K_i}^2$  where  $\{K_i\}$  are independent Poissons with mean  $\nu_i^2/2$ . Note that  $\lambda_i \geq 1$ . Then, from Oman and Zacks (1981),

$$\sum \lambda_i Y_i^2 \sim \chi_{p+2K+2L}^2 \quad (9)$$

where  $K$  is Poisson with mean  $\delta \equiv \|\nu\|^2/2$  and  $L$  is an (independent) sum of negative binomial random variables (with parameters depending on  $p$  and  $\lambda$ ). It follows that

$$1 - CP \leq P \left\{ \|\nu\|^2 \geq c^2 \chi_{p+2K}^2 \right\} \quad (10)$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \int_0^{2\delta/c^2} \frac{x^{p/2+k-1} e^{-x/2}}{\Gamma(p/2+k) 2^{p/2+k}} \frac{\delta^k e^{-\delta}}{k!} dx \\ &\leq \sum_{k=0}^{\infty} \frac{(2\delta/c^2)^{p/2+k}}{\Gamma(p/2+k) 2^{p/2+k}} \frac{\delta^k e^{-\delta}}{k!} \end{aligned} \quad (11)$$

where the last inequality uses  $e^{-x/2} \leq 1$ . Now to continue, use the fact that  $\delta^{p/2+2k} e^{-\delta}$  is maximized at  $\delta = p/2 + k$ , and use Stirling's approximation

(which is larger than the approximated  $\Gamma$ -function). Then

$$\begin{aligned}
1 - CP &\leq (1/c^2)^{p/2} \sum_{k=0}^{\infty} (1/c^2)^k \frac{(p/2 + 2k)^{p/2+2k}}{2\pi(p/2 + k)^{p/2+k+1/2} e^{-(p/2+k)} k^{k+1/2} e^{-k}} \\
&= \frac{1}{2\pi} \left(\frac{e}{c^2}\right)^{p/2} \sum_{k=0}^{\infty} \left(\frac{e^2}{c^2}\right)^k \left(1 + \frac{k}{p/2 + k}\right)^{p/2+k} \left(1 + \frac{p}{4k}\right)^k \\
&\quad \times (\max\{k, 1\}(p/2 + \max\{k, 1\}))^{-1/2} \\
&\leq \frac{1}{2\pi} \left(\frac{2e^{3/2}}{c^2}\right)^{p/2} \sum_{k=0}^{\infty} \left(\frac{2e^2}{c^2}\right)^k \frac{1}{\max\{k, 1\}} \tag{12}
\end{aligned}$$

where  $(1 + k/(p/2 + k))^{p/2+k}$  is bounded by  $2^{p/2+k}$ , and  $(1 + p/(4k))^k$  is bounded by  $e^{p/4}$ . Note that replacing  $k$  by  $\max\{k, 1\}$  follows from evaluating the summand at  $k = 0$ .

To find an explicit bound for  $c$ , use the expression  $\sum_{k=0}^{\infty} u^k / \max\{k, 1\} = 1 - \log(1 - u)$ . Convergence of the sum in (12) requires  $c^2 > 2e^2$ . So set

$$c^2 = 2e^2 \alpha^{-2/p} a \tag{13}$$

(with  $a > 0$ ), and bound  $\alpha$  by 1 when this is substituted in the log-term. Then (12) becomes

$$1 - CP \leq \alpha \left( \frac{1}{2\pi} \left( \frac{1}{a\sqrt{e}} \right)^{p/2} (1 - \log(1 - 1/a)) \right). \tag{14}$$

Thus, setting  $a = 1/(1 - \exp(-2\pi e^{p/4} + 1))$ , some algebra yields the inequality

$$1 - CP \leq \alpha a^{-p/2} \leq \alpha. \tag{15}$$

Finally, since  $p \geq 1$ , numerical evaluation gives  $a \leq 1.00086$  and one can choose  $c = 3.85 \alpha^{-1/p}$  from (13) to get uniform coverage  $1 - \alpha$ . ■

## References

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